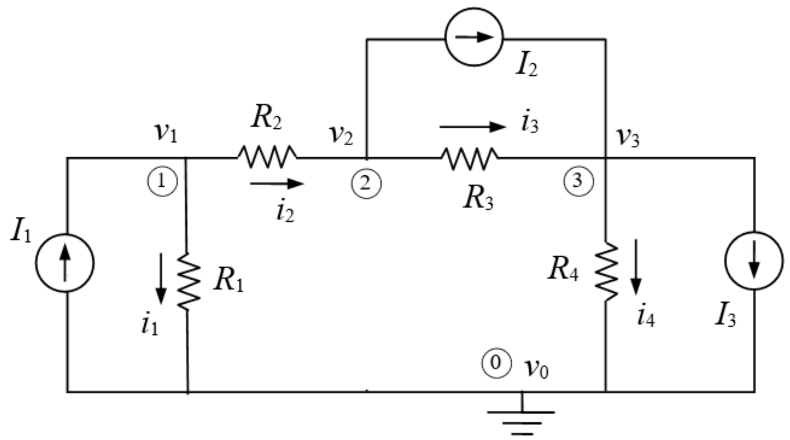
# Chap 2 Resistive Circuits

## 2.1 Node-Voltage Analysis

### 2.1.1 Independent current sources

- Resistive circuit with only independent current sources $I_1$, $I_2$ and $I_3$




Reference node: ⓪, Reference voltage: $v_0 = 0$

Nodes: ①, ②, ③, Node voltages referring to $v_0$: $v_1$, $v_2$, $v_3$

Unknown voltages to be solved: $v_1$, $v_2$, $v_3$

- Node voltage equations

KCL①: $i_1 + i_2 - I_1 = \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2} - I_1 = 0$

KCL②： $-i_2 + i_3 + I_2 = \frac{v_2 - v_1}{R_2} + \frac{v_2 - v_3}{R_3} + I_2 = 0$

KCL③： $-i_3 + i_4 - I_2 + I_3 = \frac{v_3 - v_2}{R_3} + \frac{v_3}{R_4} - I_2 + I_3 = 0$

- In terms of conductances,

KCL①: $(G_1 + G_2)v_1 - G_2 v_2 = I_1$

KCL②： $-G_2 v_1 + (G_2 + G_3)v_2 - G_3 v_3 = -I_2$

KCL③： $-G_3 v_2 + (G_3 + G_4)v_3 = I_2 - I_3$

- In matrix form,

(2.1.1-1)
$$\underbrace{\begin{bmatrix} G_1+G_2 & -G_2 & 0 \\ -G_2 & G_2+G_3 & -G_3 \\ 0 & -G_3 & G_3+G_4 \end{bmatrix}}_{\boldsymbol{A}} \cdot \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{\boldsymbol{v}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}}_{\boldsymbol{B}} \cdot \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}}_{\boldsymbol{u}}$$

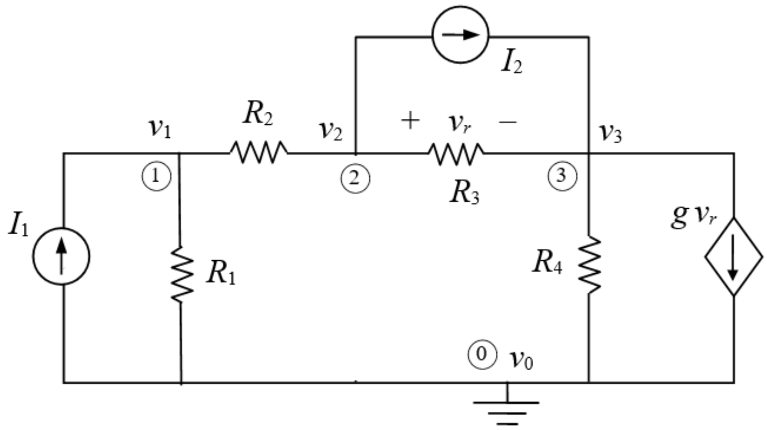
$$\Rightarrow \boldsymbol{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T = \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{u}$$

- $\boldsymbol{A}$ : symmetric positive-definite, $\boldsymbol{A}^{-1}$ exists

- All the component's voltages and currents can be obtained in terms of the node voltages. For example, $i_1 = \frac{v_1}{R_1}$, $i_2 = \frac{v_1 - v_2}{R_2}$, $i_3 = \frac{v_2 - v_3}{R_3}$, $i_4 = \frac{v_3}{R_4}$.

## 2.1.2 Dependent current sources

- Resistive circuit with only current sources including at least one dependent current source such as $gv_r$, where $v_r = v_2 - v_3$.




- Node voltage equations

KCL①: $\frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2} - I_1 = 0$

KCL②： $\frac{v_2 - v_1}{R_2} + \frac{v_2 - v_3}{R_3} + I_2 = 0$

KCL③： $-i_3 + i_4 - I_2 + gv_r = \frac{v_3 - v_2}{R_3} + \frac{v_3}{R_4} - I_2 + g\left(v_2 - v_3\right) = 0$

- In terms of conductances,

KCL①: $(G_1+G_2)v_1 - G_2 v_2 = I_1$

KCL②: $-G_2 v_1 + (G_2+G_3)v_2 - G_3 v_3 = -I_2$

KCL③: $(-G_3+g)v_2 + (G_3+G_4-g)v_3 = I_2$

- In matrix form,

(2.1.2-1) $$\underbrace{\begin{bmatrix} G_1+G_2 & -G_2 & 0 \\ -G_2 & G_2+G_3 & -G_3 \\ 0 & -G_3+g & G_3+G_4-g \end{bmatrix}}_{\boldsymbol{A}} \cdot \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{\boldsymbol{v}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}}_{\boldsymbol{B}} \cdot \underbrace{\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}}_{\boldsymbol{u}}$$

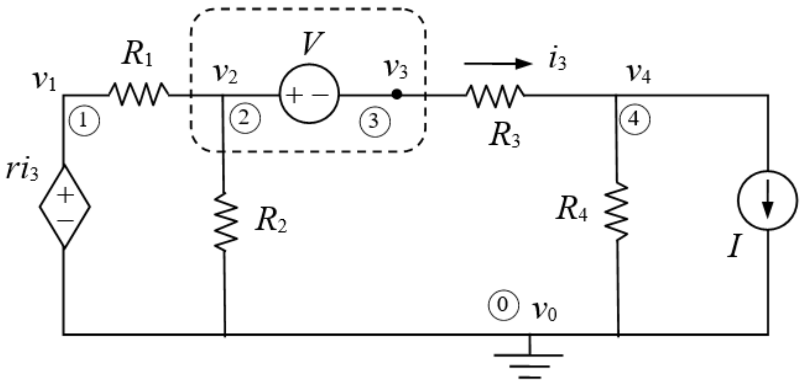
- If $g \neq G_3 + G_4 + G_3 G_4 \dfrac{G_1+G_2}{G_1 G_2}$, then $\boldsymbol{A}$ is invertible and $\boldsymbol{v} = \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{u}$.

- All the component's voltages and currents in the resistive circuit can be expressed in terms of the node voltages.

### 2.1.3 Voltage sources

- Resistive circuit icluding at least one voltage source such as independent voltage source $V$ and dependent voltage source $ri_3$, where $i_3 = \dfrac{v_3 - v_4}{R_3}$.

- Supernode'②③'.




- Node voltage equations

KVL①: $v_1 = ri_3 = r\left(\dfrac{v_3 - v_4}{R_3}\right)$

KCL②③： $\frac{v_2-v_1}{R_1}+\frac{v_2}{R_2}+\frac{v_3-v_4}{R_3}=0$

KVL②③： $v_2-v_3=V$

KCL④： $\frac{v_4-v_3}{R_3}+\frac{v_4}{R_4}+I=0$

- In terms of conductances,

KVL①: $v_1-rG_3v_3+rG_3v_4=0$

KCL②③： $-G_1v_1+(G_1+G_2)v_2+G_3v_3-G_3v_4=0$

KVL②③： $v_2-v_3=V$

KCL④： $-G_3v_3+(G_3+G_4)v_4=-I$

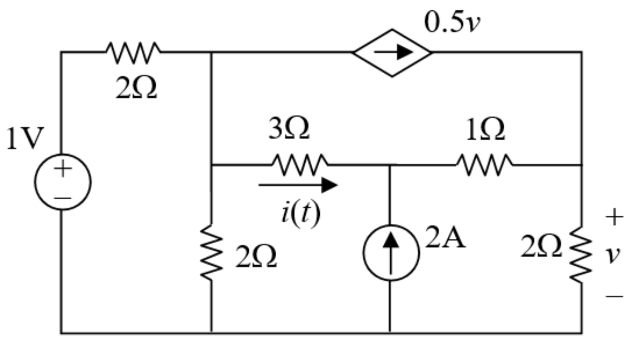
- In matrix form,

(2.1.3-1) $$\underbrace{\begin{bmatrix} 1 & 0 & -rG_3 & rG_3 \\ -G_1 & G_1+G_2 & G_3 & -G_3 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -G_3 & G_3+G_4 \end{bmatrix}}_{\boldsymbol{A}} \cdot \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}}_{\boldsymbol{v}} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\boldsymbol{B}} \cdot \underbrace{\begin{bmatrix} V \\ I \end{bmatrix}}_{\boldsymbol{u}}$$

- If $r \neq \frac{1}{G_1}+\left(\frac{1}{G_3}+\frac{1}{G_4}\right)\left(1+\frac{G_2}{G_1}\right)$, then $\boldsymbol{A}$ is invertible and $\boldsymbol{v}=\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{u}$.

- All the component's voltages and currents in the resistive circuit can be expressed in terms of the node voltages.

Example: Based on node voltage analysis, determine $i(t)$.

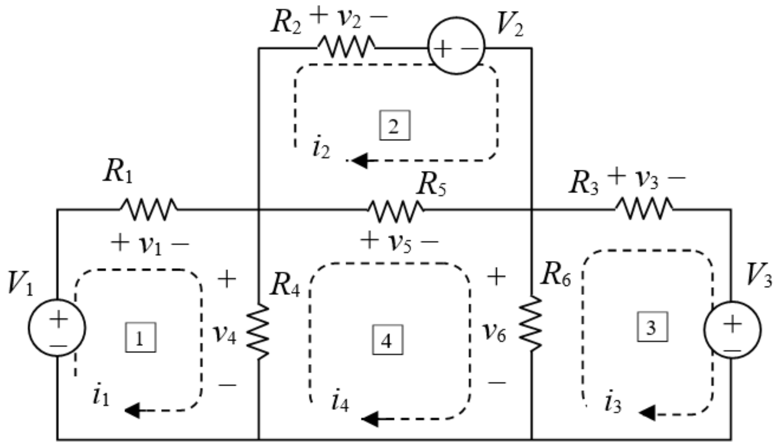


## 2.2 Mesh-Current Analysis

- Unlike the node-voltage analysis, the mesh-current analysis is only suitable for planar circuit.

### 2.2.1 Independent voltage sources

- Resistive circuit with only independent voltage sources $V_1$, $V_2$ and $V_3$




Meshes: 1, 2, 3, 4,     Mesh currents: $i_1$, $i_2$, $i_3$, $i_4$

Unknown currents to be solved: $i_1$, $i_2$, $i_3$, $i_4$

- Mesh current equations

KVL1: $v_1 + v_4 - V_1 = R_1 i_1 + R_4 (i_1 - i_4) - V_1 = 0$

KVL2 : $v_2 + V_2 - v_5 = R_2 i_2 + V_2 - R_5 (i_4 - i_2) = 0$

KVL3 : $v_3 + V_3 - v_6 = R_3 i_3 + V_3 - R_6 (i_4 - i_3) = 0$

KVL4 : $v_5 + v_6 - v_4 = R_5 (i_4 - i_2) + R_6 (i_4 - i_3) - R_4 (i_1 - i_4) = 0$

- In terms of resistance,

KVL1: $(R_1 + R_4) i_1 - R_4 i_4 = V_1$

KVL2 : $(R_2 + R_5) i_2 - R_5 i_4 = -V_2$

KVL3 : $(R_3 + R_6) i_3 - R_6 i_4 = -V_3$

KVL4 : $-R_4 i_1 - R_5 i_2 - R_6 i_3 + (R_4 + R_5 + R_6) i_4 = 0$

- In matrix form,

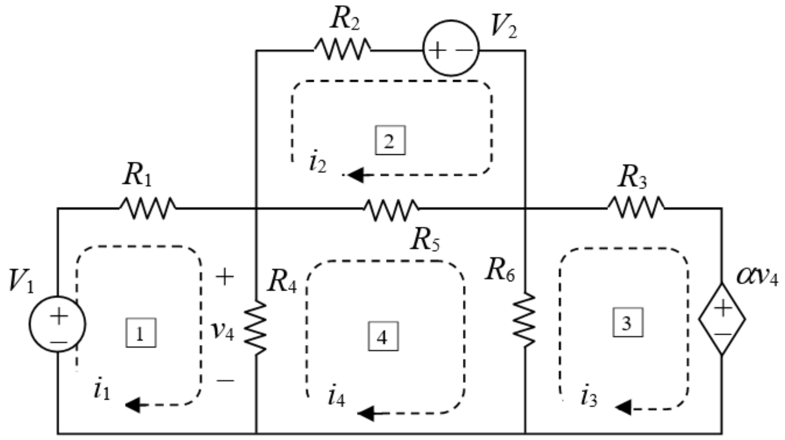
$$(2.2.1\text{-}1)\quad \underbrace{\begin{bmatrix} R_1+R_4 & 0 & 0 & -R_4 \\ 0 & R_2+R_5 & 0 & -R_5 \\ 0 & 0 & R_3+R_6 & -R_6 \\ -R_4 & -R_5 & -R_6 & R_4+R_5+R_6 \end{bmatrix}}_{\boldsymbol{A}} \cdot \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}}_{\boldsymbol{i}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{B}} \cdot \underbrace{\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}}_{\boldsymbol{u}}$$

$$\Rightarrow\ \boldsymbol{i} = \begin{bmatrix} i_1 & i_2 & i_3 & i_4 \end{bmatrix}^T = \boldsymbol{A}^{-1}\boldsymbol{Bu}$$

- $\boldsymbol{A}$ is symmetric positive-definite, $\boldsymbol{A}^{-1}$ exists.
- All the component's voltages and currents can be obtained in terms of the mesh currents. For example, $v_1 = R_1 i_1$, $v_3 = R_3 i_3$, $v_5 = R_5(i_4 - i_2)$, $v_6 = R_6(i_4 - i_3)$.

## 2.2.2 Dependent voltage sources

- Resistive circuit with only voltage sources including at least one dependent voltage source, such as $\alpha v_4$, where $v_4 = R_4(i_1 - i_4)$.




- Based on KVL, mesh current equations are expressed as

KVL[1]: $R_1 i_1 + R_4(i_1 - i_4) - V_1 = 0$

KVL[2]: $R_2 i_2 + V_2 + R_5(i_2 - i_4) = 0$

KVL[3]: $R_3 i_3 + \alpha v_4 + R_6(i_3 - i_4) = R_3 i_3 + \alpha R_4(i_1 - i_4) + R_6(i_3 - i_4) = 0$

KVL[4]: $R_5(i_4 - i_2) + R_6(i_4 - i_3) + R_4(i_4 - i_1) = 0$

- In terms of resistance,

KVL[1]: $(R_1+R_4)i_1 - R_4 i_4 = V_1$

KVL[2]: $(R_2+R_5)i_2 - R_5 i_4 = -V_2$

KVL[3]: $\alpha R_4 i_1 + (R_3+R_6)i_3 - (\alpha R_4 + R_6)i_4 = 0$

KVL[4]: $-R_4 i_1 - R_5 i_2 - R_6 i_3 + (R_4+R_5+R_6)i_4 = 0$

- In matrix form,

(2.2.2-1)
$$\underbrace{\begin{bmatrix} R_1+R_4 & 0 & 0 & -R_4 \\ 0 & R_2+R_5 & 0 & -R_5 \\ \alpha R_4 & 0 & R_3+R_6 & -\alpha R_4 - R_6 \\ -R_4 & -R_5 & -R_6 & R_4+R_5+R_6 \end{bmatrix}}_{\boldsymbol{A}} \cdot \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}}_{\boldsymbol{i}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\boldsymbol{B}} \cdot \underbrace{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}}_{\boldsymbol{u}}$$

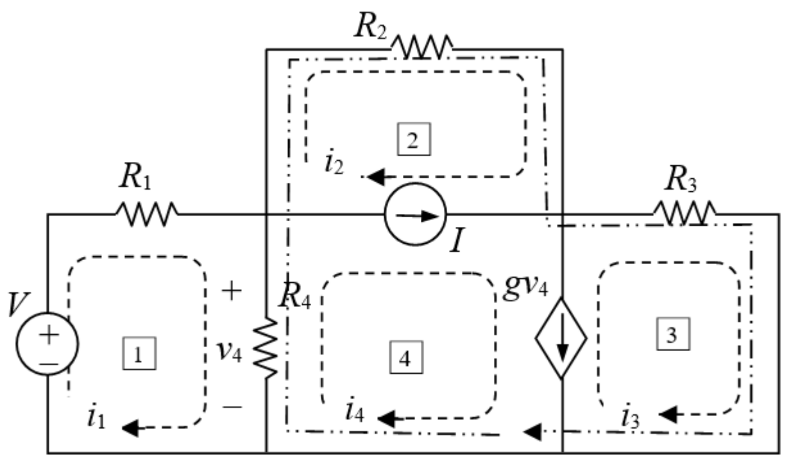
- If $\alpha \neq 1 + \frac{R_3}{R_6} + R_3\left(\frac{1}{R_1} + \frac{1}{R_4}\right) + \frac{R_2 R_5}{R_2+R_5}\left(1+\frac{R_3}{R_6}\right)\left(\frac{1}{R_1}+\frac{1}{R_4}\right)$, then $\boldsymbol{A}^{-1}$ exists and

(2.2.2-2) $\boldsymbol{i} = \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{u}$

- All the component's voltages and currents in the resistive circuit can be obtained in terms of the mesh currents.

### 2.2.3 Current sources

- Resistive circuit icluding at least one current source such as independent current source $I$ and dependent current source $g v_4$, where $v_4 = R_4(i_1 - i_4)$.




- Supermesh '[2][3][4]'.

- Mesh current equations

KVL$\boxed{1}$: $R_1 i_1 + R_4 \left(i_1 - i_4\right) - V = 0$

KVL$\boxed{2}\boxed{3}\boxed{4}$ : $R_2 i_2 + R_3 i_3 + R_4 \left(i_4 - i_1\right) = 0$

KCL$\boxed{2}\boxed{3}\boxed{4}$ : $i_3 - i_4 = -g v_4 = -g R_4 \left(i_1 - i_4\right)$

KCL$\boxed{2}\boxed{3}\boxed{4}$ : $i_4 - i_2 = I$

- In terms of resistance,

KVL$\boxed{1}$: $\left(R_1 + R_4\right) i_1 - R_4 i_4 = V$

KVL$\boxed{2}\boxed{3}\boxed{4}$ : $-R_4 i_1 + R_2 i_2 + R_3 i_3 + R_4 i_4 = 0$

KCL$\boxed{2}\boxed{3}\boxed{4}$ : $g R_4 i_1 + i_3 - \left(1 + g R_4\right) i_4 = 0$

KCL$\boxed{2}\boxed{3}\boxed{4}$ : $-i_2 + i_4 = I$

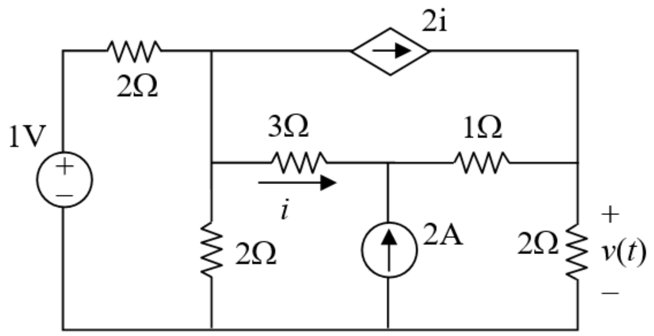
- In matrix form,

$$\underbrace{\begin{bmatrix} R_1 + R_4 & 0 & 0 & -R_4 \\ -R_4 & R_2 & R_3 & R_4 \\ g R_4 & 0 & 1 & -1 - g R_4 \\ 0 & -1 & 0 & 1 \end{bmatrix}}_{\boldsymbol{A}} \cdot \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}}_{\boldsymbol{i}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\boldsymbol{B}} \cdot \underbrace{\begin{bmatrix} V \\ I \end{bmatrix}}_{\boldsymbol{u}} \tag{2.2.3-1}$$

Since $g \neq -\left(\frac{1}{R_1} + \frac{1}{R_4}\right)\left(1 + \frac{R_2}{R_3}\right) - \frac{1}{R_3}$, $\boldsymbol{A}$ is invertible and $\boldsymbol{i} = \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{u}$.

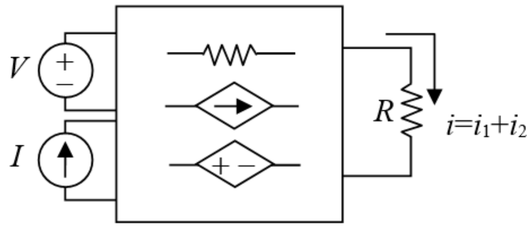
- All the component's voltages and currents in the resistive circuit can be expressed in terms of the mesh currents.

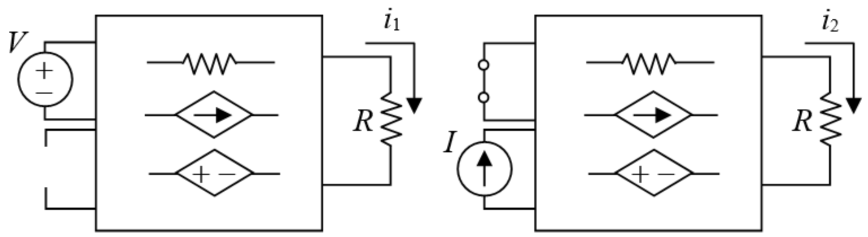
Example: Based on mesh current analysis, determine $v(t)$.



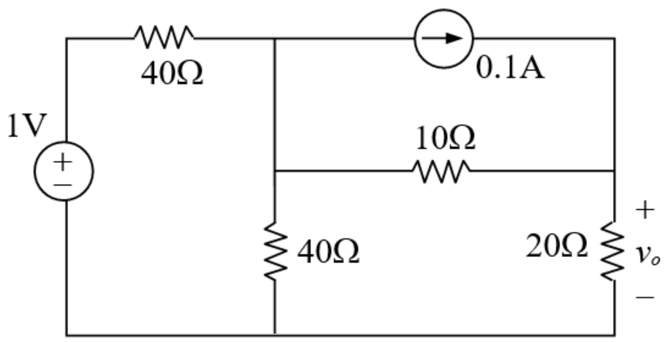
## 2.3 Principle of Superposition

- Resistive circuit with independent voltage source *V* and current source *I*



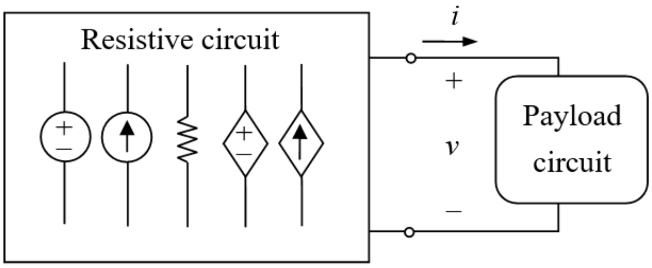



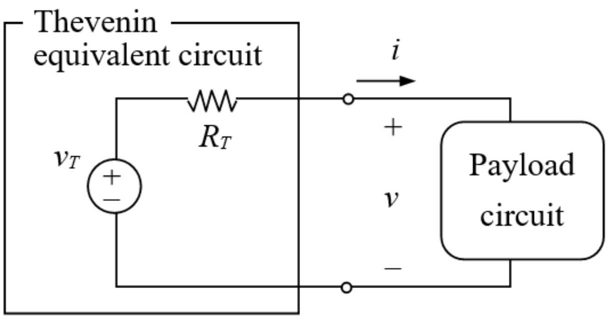

Example: Determine $v_o$.



## 2.4 Thevenin Equivalent Circuits

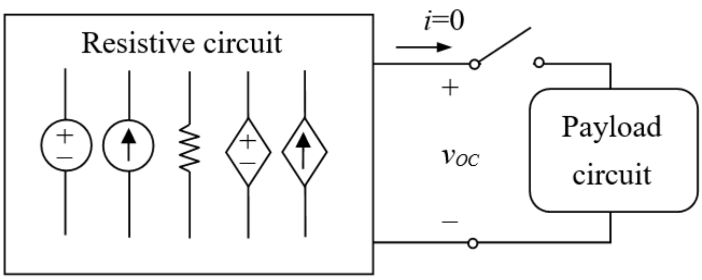
- Resistive circuit with Indeoendent sources



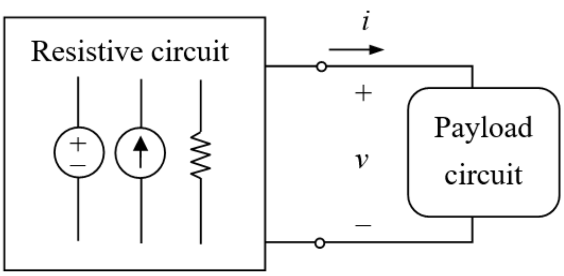

(2.4-1) $v_T = v + R_T i$

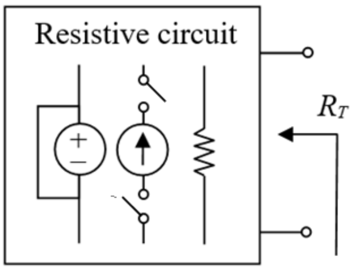
- Thevenin equivalent voltage $v_T = v_{OC}$



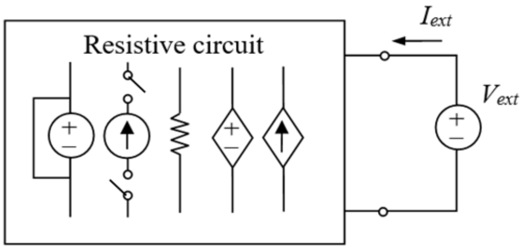

- Thevenin equivalent resistance $R_T$

[A] Without dependent sources



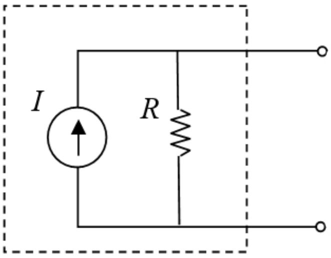



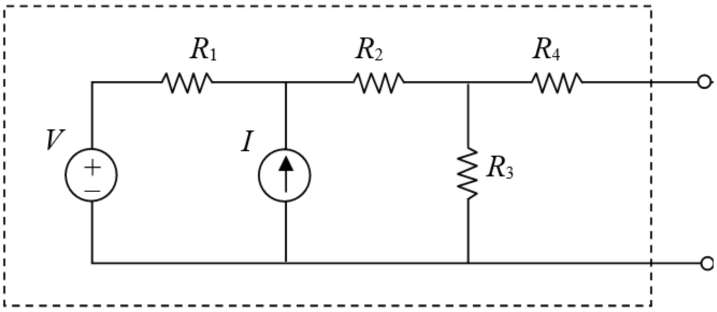
[B] With dependent sources

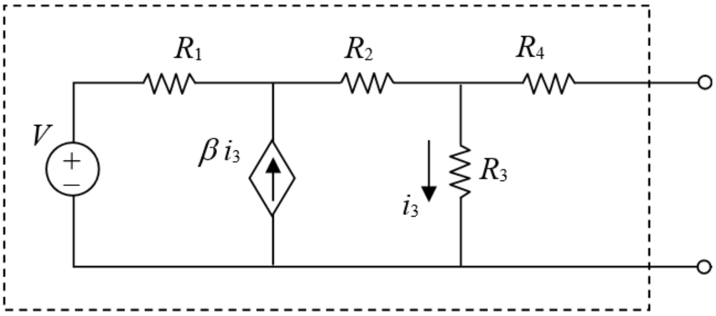



(2.4-2) $$R_T = \frac{V_{ext}}{I_{ext}}$$

<u>Example</u>: Determine the Thevenin equivalent circuit.

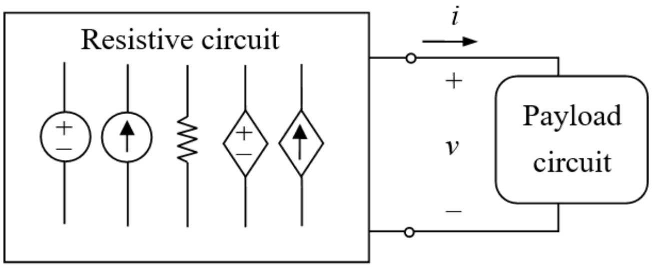


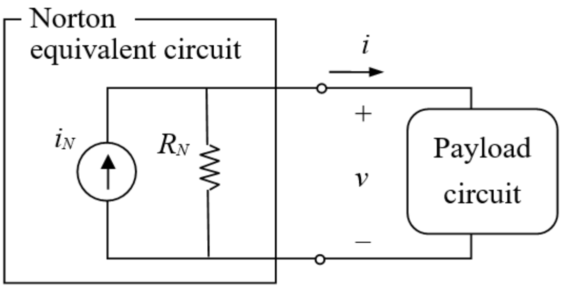



## 2.5 Norton Equivalent Circuits

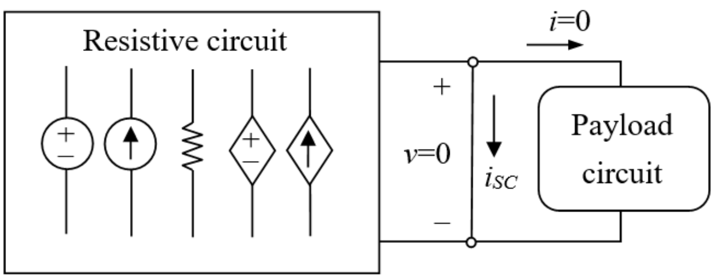
- Resistive circuit with independent sources



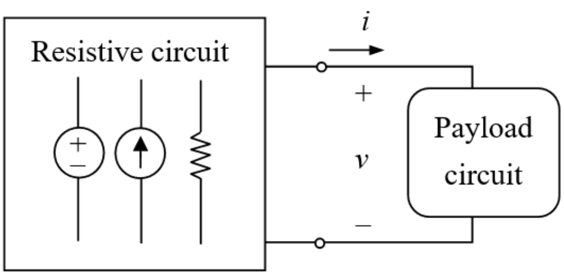

(2.5-1) $$i_N = i + \frac{v}{R_N}$$

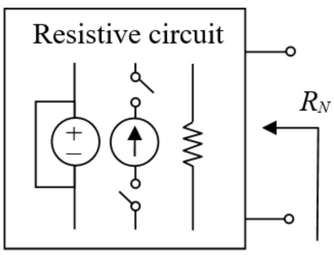
- Norton equivalent current $i_N = i_{SC}$



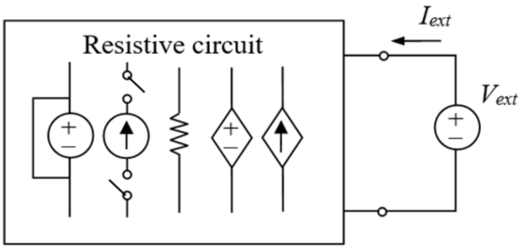

- Norton equivalent resistance $R_N$

[A] Without dependent sources



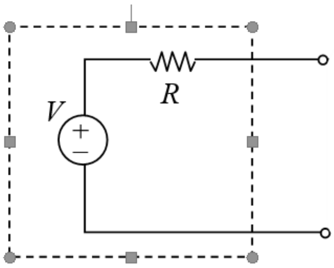



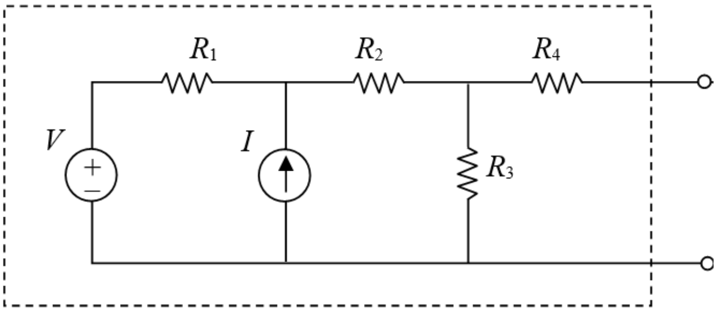
[B] With dependent sources

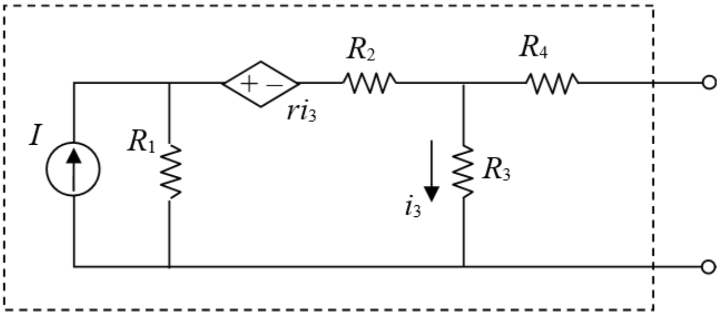



(2.5-2) $$R_N = \frac{V_{ext}}{I_{ext}}$$

Example: Determine the Norton equivalent circuit.

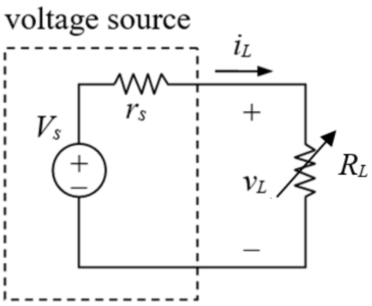


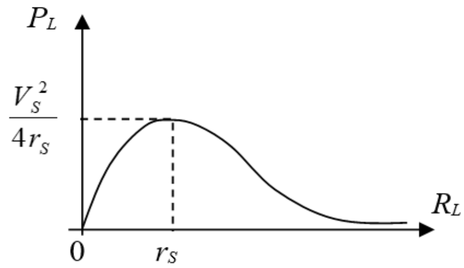



## 2.6 Maximum Power Transfer Theorem

- The power transferred from voltage source to adjustable $R_L$




(2.6-1) $$i_L = \frac{V_S}{r_S + R_L}, \qquad v_L = R_L i_L = \frac{R_L V_S(t)}{r_S + R_L}$$

The power provided by voltage source

(2.6-2) $$P_S = V_S\, i_L = \frac{V_S^2}{r_S + R_L}$$

The power transferred to $R_L$

(2.6-3) $$P_L = i_L v_L = \frac{R_L V_S^2(t)}{(r_S + R_L)^2} = \frac{R_L}{r_S + R_L} P_S$$

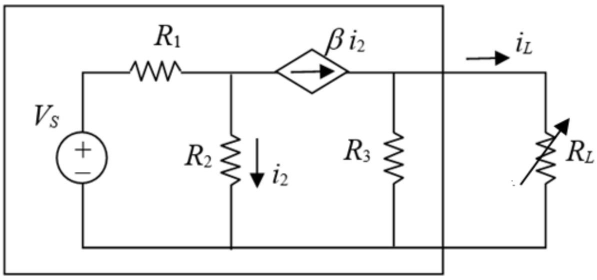
The derivatives of $P_L$ with respect to $R_L$ are

$$\frac{dP_L}{dR_L} = \frac{(r_S - R_L)V_S^2(t)}{(r_S + R_L)^3} \quad \text{and} \quad \frac{d^2P_L}{dR_L{}^2} = \frac{-2(2r_S - R_L)V_S^2(t)}{(r_S + R_L)^4}$$

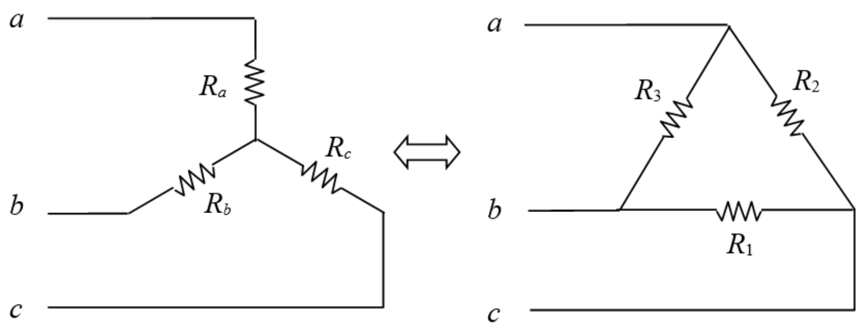
Since $\left.\frac{dP_L}{dR_L}\right|_{R_L = r_S} = 0$ and $\left.\frac{d^2P_L}{dR_L{}^2}\right|_{R_L = r_S} < 0$, the maximum transferred power is

(2.6-4) $$P_{L,max} = P_L\big|_{R_L = r_S} = \frac{V_S^2(t)}{4r_S} = \frac{1}{2} P_S\big|_{R_L = r_S}$$

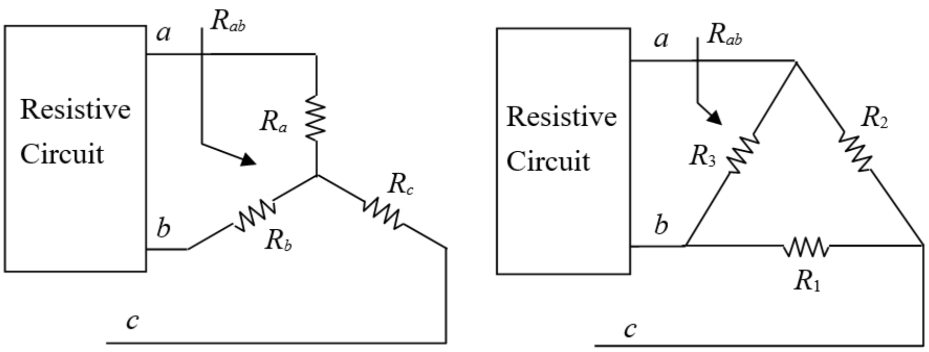
Example: Determine the maximum transferred power to adjustable $R_L$.



## 2.7 Y-Δ Resistive Circuit Transformation




- Resistance $R_{ab}$ with terminal $c$ open




(2.7-1) $$R_a + R_b = R_3 \parallel (R_1 + R_2) = \frac{R_3(R_1+R_2)}{R_1+R_2+R_3} = \frac{R_2R_3+R_3R_1}{R_1+R_2+R_3}$$

- The relation between Y-type and Δ-type resistive circuits

(2.7-2) $$\begin{cases} R_a + R_b = \dfrac{R_3(R_1+R_2)}{R_1+R_2+R_3} \\ R_b + R_c = \dfrac{R_1(R_2+R_3)}{R_1+R_2+R_3} \\ R_c + R_a = \dfrac{R_2(R_3+R_1)}{R_1+R_2+R_3} \end{cases} \Rightarrow R_a + R_b + R_c = \frac{R_1R_2+R_2R_3+R_3R_1}{R_1+R_2+R_3}$$

Hence,

(2.7-3) $$R_a = \frac{R_2R_3}{R_1+R_2+R_3}, \quad R_b = \frac{R_3R_1}{R_1+R_2+R_3}, \quad R_c = \frac{R_1R_2}{R_1+R_2+R_3}$$

which results in

(2.7-4) $$R_1R_a = R_2R_b = R_3R_c = \frac{R_1R_2R_3}{R_1+R_2+R_3}$$

(2.7-5)
$$\begin{cases} R_a R_b = \dfrac{R_3\left(R_1R_2R_3\right)}{\left(R_1+R_2+R_3\right)^2} \\ R_b R_c = \dfrac{R_1\left(R_1R_2R_3\right)}{\left(R_1+R_2+R_3\right)^2} \\ R_c R_a = \dfrac{R_2\left(R_1R_2R_3\right)}{\left(R_1+R_2+R_3\right)^2} \end{cases} \Rightarrow R_aR_b+R_bR_c+R_cR_a = \frac{R_1R_2R_3}{R_1+R_2+R_3}$$

From (2.7-4) and (2.7-5), we have

(2.7-6) $$R_1R_a = R_2R_b = R_3R_c = R_aR_b+R_bR_c+R_cR_a$$

which leads to

(2.7-7) $$R_1 = \frac{R_aR_b+R_bR_c+R_cR_a}{R_a}$$

(2.7-8) $$R_2 = \frac{R_aR_b+R_bR_c+R_cR_a}{R_b}$$

(2.7-9) $$R_3 = \frac{R_aR_b+R_bR_c+R_cR_a}{R_c}$$

$$\underbrace{\frac{R_1R_2R_3}{R_1+R_2+R_3} = R_1R_a = R_2R_b = R_3R_c}_{\text{from }\Delta\text{ to Y}} \!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\! \overbrace{\phantom{=}R_1R_a = R_2R_b = R_3R_c = R_aR_b+R_bR_c+R_cR_a}^{\text{from Y to }\Delta}$$

Example: Determine $v(t)$.